The Logarithmic Sobolev Inequality Along The Ricci Flow: The Case $\lambda_0(g_0) = 0$

Rugang Ye
Department of Mathematics
University of California, Santa Barbara

August 1, 2007

1 Introduction

In [Y1] and [Y2], logarithmic Sobolev inequalities along the Ricci flow in all dimensions $n \geq 2$ were obtained using Perelman's entropy monotonicity, which lead to Sobolev inequalities and κ-noncollpasing estimates. In particular, a uniform logarithmic Sobolev inequality, a uniform Sobolev inequality and a uniform κ-noncollapsing estimate were obtained without any restriction on time, provided that the smallest eigenvalue $\lambda_0(g_0)$ of the operator $-\Delta + \frac{R}{4}$ for the initial metric is positive. In this paper, we extend these uniform results to the case $\lambda_0(g_0) = 0$.

Consider a compact manifold M of dimension $n \geq 2$. Let $g = g(t)$ be a smooth solution of the Ricci flow

$$\frac{\partial g}{\partial t} = -2Ric \tag{1.1}$$

on $M \times [0, T)$ for some (finite or infinite) $T > 0$ with a given initial metric $g(0) = g_0$.

Theorem A *Assume that $\lambda_0(g_0) = 0$. For each $t \in [0, T)$ and each $\sigma > 0$ there holds*

$$\int_M u^2 \ln u^2 dvol \leq \sigma \int_M (|\nabla u|^2 + \frac{R}{4} u^2) dvol - \frac{n}{2} \ln \sigma + C \tag{1.2}$$

for all $u \in W^{1,2}(M)$ with $\int_M u^2 dvol = 1$, where C depends only on (M, g_0).

Note that as in [Y1] a log gradient version of the logarithmic Sobolev inequality follows as a consequence of (1.2). We omit its statement. Our next result provides a dependence of the above C on g_0 in terms of rudimentary geometric data. To simplifiy the statements, we assume that $|Rm| \leq 1$ for g_0, which can always be achieved

by a rescaling. Then we can also assume $T \geq 2\alpha(n)$ for a positive constant $\alpha(n)$ depending only on n such that $|Rm| \leq 2$ on $[0, \alpha(n)]$. (Namely the maximal possible T such that the solution $g = g(t)$ can be extended to a smooth solution of the Ricci flow on $[0, T)$ has this property.)

Theorem B *There are for each $v_0 > 0$, each $D_0 > 0$, each $\epsilon > 0$ and each integer $l \geq 3$ a positive number $C = C(v_0, D_0, \epsilon, l, n)$ with the following properties. Assume $\lambda_0(g_0) = 0, vol_{g_0}(M) \geq v_0, diam_{g_0} \leq D_0$ and the normalization conditions $|Rm|_{g_0} \leq 1$ and $T \geq 2\alpha(n)$. $vol_{g_0}(M) \geq v_0$. Then one of the following two cases must occur:*
1) $g(\alpha(n))$ lies in the ϵ-neighborhood of a Ricci flat metric on M in the C^l norm,
2) the logarithmic Sobolev inequality (1.2) holds true for each $t \in [0, T)$, each $\sigma > 0$ and all $u \in W^{1,2}(M)$ with $\int_M u^2 dvol = 1$, where $C = C(v_0, D_0, \epsilon, l, n)$.

It turns out that we have a better result in dimension $n = 3$. The same holds true in dimension $n = 2$. But Theorem E and Theorem 3.7 in [Y2] provide a stronger result in this dimension.

Theorem C *Assume that $n = 3$. There is for each $v_0 > 0$ and each $D_0 > 0$ a positive number $C = C(v_0, D_0)$ with the following properties. Assume $\lambda_0(g_0) = 0, vol_{g_0}(M) \geq v_0, diam_{g_0} \leq D_0$ and the normalization conditions $|Rm|_{g_0} \leq 1$ and $T \geq 2\alpha(n)$. Then the logarithmic Sobolev inequality (1.2) holds true for each $t \in [0, T)$, each $\sigma > 0$ and all $u \in W^{1,2}(M)$ with $\int_M u^2 dvol = 1$, where $C = C(v_0, D_0)$.*

As in [Y1], Theorem A, Theorem B and Theorem C lead to Sobolev inequalities along the Ricci flow, which in turn lead to κ-noncollpasing estimates. We consider only the case $n \geq 3$ although the methods also work for $n = 2$, because the case $n = 2$ is covered by the results in [Y2].

Theorem D *Assume that $n \geq 3$ and $\lambda_0(g_0) = 0$. Then there holds for each $t \in [0, T)$*

$$\left(\int_M |u|^{\frac{2n}{n-2}} dvol \right)^{\frac{n-2}{n}} \leq A \int_M (|\nabla u|^2 + \frac{R}{4} u^2) dvol + B \int_M u^2 dvol \tag{1.3}$$

for all $u \in W^{1,2}(M)$, where A and B depend only on (M, g_0). We have $A = A(v_0, D_0)$ and $B = B(v_0, D_0)$ for given $v_0 > 0$ and $D_0 > 0$, if $n = 3$ and g_0 satisfies the conditions in Theorem C. We also have $A = A(v_0, D_0, n)$ and $B = B(v_0, D_0, n)$ for given $v_0 > 0$ and $D_0 > 0$, provided that g_0 satisfies the conditions in Theorem B and $g(\alpha(n))$ does not lie in the ϵ-neighborhood of any Ricci flat metric on M in the C^3 norm.

Theorem E *Assume that $n = 3$ and $\lambda_0(g_0) = 0$. Let $L > 0$ and $t \in [0, T)$. Consider the Riemannian manifold (M, g) with $g = g(t)$. Assume $R \leq \frac{1}{r^2}$ on a geodesic ball*

$B(x,r)$ with $0 < r \le L$. Then there holds

$$vol(B(x,r)) \ge \left(\frac{1}{2^{n+3}A + 2BL^2}\right)^{\frac{n}{2}} r^n, \tag{1.4}$$

where A and B are from Theorem D.

As in [Y1], the above results extend to various versions of the modified Ricci flows. Moreover, the κ-noncollapsing estimates ensure that we can obtain smooth blow-up limits at the time infinity under the assumption that $\lambda_0(g_0) = 0$. We omit the statements of those results because they are completely analogous to the corresponding ones in [Y1].

2 The Proofs

Proof of Theorem A Consider a fixed $t_1 \in (0, T)$. Let u_1 be a positive eigenfunction for the eigenvalue $\lambda_0(g(t_1))$ associated with the metric $g(t_1)$, such that $\int_M u_1^2 dvol = 1$ with respect to $g(t_1)$. Let $f = f(t)$ be the smooth solution of the equation

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R \tag{2.1}$$

on $[0, t_1]$ with $f(t_1) = -2 \ln u_1$. Note that (2.1) is equivalent to

$$\frac{\partial v}{\partial t} = -\Delta v + Rv, \tag{2.2}$$

where $v = e^{-f}$. So the solution $f(t)$ exists. We also infer $\frac{d}{dt}\int_M vdvol = 0$, and hence $\int_M vdvol = 1$ for all $t \in [0, t_1]$.

We set $u = e^{-\frac{f}{2}}$. By [P, (1.4)] we then have

$$\frac{d}{dt}\int_M (|\nabla u|^2 + \frac{R}{4}u^2)dvol = \frac{1}{4}\frac{d}{dt}\int_M (|\nabla f|^2 + R)e^{-f}dvol \ge \frac{1}{2}\int_M |Ric + \nabla^2 f|^2 e^{-f}dvol. \tag{2.3}$$

It follows that

$$\lambda_0(g(t_1)) \ge \lambda_0(g_0) + \frac{1}{2}\int_0^{t_1}\int_M |Ric + \nabla^2 f|^2 e^{-f} dvoldt. \tag{2.4}$$

We choose $t_1 = \min\{\frac{T}{2}, 1\}$. If $\lambda_0(g(t_1) > 0$, we first apply Theorem A in [Y1] or Theorem A in [Y2] to obtain (1.2) for $g(t)$ on $[0, t_1]$. Then we apply Theorem 4.2 in [Y1] to obtain (1.2) on $[t_1, T)$ with a larger C. If $\lambda_0(g(t_1)) = 0$, we deduce from (2.4)

$$Ric + \nabla^2 f = 0 \tag{2.5}$$

on $[0, t_1]$. It follows that $g = g(t)$ is a steady Ricci soliton for all t. Hence the logarithmic Sobolev inequality for g_0 provided by Theorem 3.3 in [Y1] holds true for all $g(t)$. Actually, [CK, Proposition 5.20] implies that g_0 is Ricci flat, hence $g(t) = g_0$ for all t. ■

Lemma 2.1 *For given $v_0 > 0, D_0 > 0, \epsilon > 0$ and $l \geq 3$ there is a positive constant $\mu_0 = \mu_0(v_0, D_0, \epsilon, l, n)$ with the following properties. Let $g = g(t)$ be a smooth solution of the Ricci flow on $M \times [0, T)$ which satisfies the normalization conditions in Theorem B and the condtions $vol_{g(0)} \geq v_0$ and $diam_{g(0)} \leq D_0$. Assume $\lambda_0(g_0) = 0$. If $\lambda(g(\alpha(n))) < \mu_0$, then $g(\alpha(n))$ lies in the ϵ-neighborhood of a Ricci flat metric with respect to the C^l-norm.*

Proof. Assume that μ_0 does not exist. Then we can find a sequence of manifolds M_k of a fixed dimension n and a sequence of smooth solutions $g_k = g_k(t)$ on $M_k \times [0, T_k)$ satisfying the normalization conditions and the conditions $vol_{g_k(0)} \geq v_0$ and $diam_{g_k(0)} \leq D_0$, such that $\lambda_0(g_k(0)) = 0$, $\lambda_0(g_k(\alpha(n))) \to 0$, and $g_k(\alpha(n))$ does not lie in the ϵ-neighborhood of any Ricci flat metric with respect to the C^l-norm. By Gromov-Cheeger-Hamilton compactness theorem [H], we can find a subsequence of (M_k, g_k), which we still denote by (M_k, g_k), such that $(M_k, g_k, [\frac{\alpha(n)}{2}, \alpha(n)])$ converge smoothly to a limit Ricci flow $(M, g, [\frac{\alpha(n)}{2}, \alpha(n)])$. There holds $\lambda_0(g(\alpha(n))) = 0$. By the monotonicity of λ_0 along the Ricci flow (see [P] or the above proof of Theorem A), we have $\lambda_0(g_k(t)) \geq 0$ for all $t \in [0, T_k)$. Hence $\lambda_0(g(t)) \geq 0$ for all $t \in [\frac{\alpha(n)}{2}, \alpha(n)]$. Now the argument in the proof of Theorem A implies that $g(\alpha(n))$ is Ricci flat. But $(M_k, g_k(\alpha(n)))$ converge smoothly to $(M, g(\alpha(n))$, so $g_k(\alpha(n))$ lies in the ϵ-neighborhood of a Ricci flat metric with respect to the C^l-norm whenever k is large enough. This is a contradiction. ■

Proof of Theorem B Assume that $g(\alpha(n))$ does not lie in the ϵ-neighborhood of any Ricci flat metric with respect to the C^l-norm. Then $\lambda_0(g(\alpha(n))) \geq \mu_0$ by Lemma 2.1. Now we obtain a desired logarithmic Sobolev inequality for $g(t)$ on $[0, \alpha(n)]$ by Theorem A in [Y1]. Alternatively, we can apply the arguments in [Y3] for controlling the evolution of the Sobolev constant to bound the Sobolev constant for $g(t)$ on $[0, \alpha(n)]$, and then apply Theorem 3.3 in [Y1] to infer the desired logarithmic Sobolev inequality. Next we apply the bound for the Sobolev constant at $t = \alpha(n)$, the bound $\lambda_0 \geq \mu_0$ at $t = \alpha(n)$ and the arguments in the proof of Theorem 3.5 in [Y1] to deduce a logarithmic Sobolev inequality of the kind [Y1, (3.11)] at $t = \alpha(n)$. Then we apply Theorem B in [Y1] on $[\alpha(n), T)$ and combine it with Theorem A in [Y1]. Then we arrive at the desired logarithmic Sobolev inequality on $[\alpha(n), T)$. ■

Proof of Theorem C By [GIK], there is for each given flat metric g an ϵ-neighbord of g with respect to the C^6-norm, such that the Ricci flow starting at any metric in

the neighborhood converges smoothly to a Ricci flat metric at a fixed exponential rate as $t \to \infty$. Moreover, the limit Ricci flat metric lies in the 2ϵ-neighborhood of g with respect to the C^6-norm. We call such a neighborhood a *Ricci contraction ϵ-neighborhood*.

Now for given $v_0 > 0, D_0 > 0$ and $K_0 > 0$ the moduli space $\mathcal{M}^0(v_0, D_0, K_0)$ of flat metrics on M with $vol \geq v_0, diam \leq D_0$ and $|Rm| \leq K_0$ is C^∞ compact modulo diffeomorphisms by Gromov-Cheeger compactness theorem and the Einstein equation. So there is a uniform $\epsilon = \epsilon(v_0, D_0, K_0)$ such that each $g \in \mathcal{M}^0(v_0, D_0, K_0)$ has a Ricci contraction ϵ-neighborhood. Moreover, there is a unform upper bound $C(v_0, D_0, K_0)$ for the Sobolev constant for $g \in \mathcal{M}^0(v_0, D_0, K_0)$.

Now consider for given $v_0 > 0$ and $D_0 > 0$ a smooth solution of the Ricci flow $g = g(t)$ satisfying the normalization conditions and the conditions $vol_{g_0}(M) \geq v_0$ and $diam_{g_0}(M) \leq D_0$. Let $\epsilon > 0$ and $l = 6$, where ϵ is to be determined. Assume that $g(\alpha(n))$ lies in the ϵ-neighborhood of a Ricci flat metric $\bar{g}$ with respect to the C^6 norm. Since $n = 3$, $\bar{g}$ is flat. There is a positive number $\epsilon_0 = \epsilon_0(v_0, D_0)$ such that $vol_{\bar{g}}(M) \geq \frac{1}{2}v_0, diam_{\bar{g}}(M) \leq 2D_0$ and $|Rm|_{\bar{g}} \leq 3$, i.e. $\bar{g} \in \mathcal{M}^0(\frac{1}{2}v_0, 2D_0, 3)$, whenever $\epsilon \leq \epsilon_0$. Now we choose $\epsilon = \min\{\epsilon_0(v_0, D_0), \epsilon(\frac{1}{2}v_0, 2D_0, 4)\}$. Then the maximally extended $g(t)$ converges smoothly to a flat metric g^* at exponential rate where the rate depends only on v_0 and D_0. We can choose ϵ_0 sufficiently small such that $g^* \in \mathcal{M}^0(\frac{1}{3}v_0, 3D_0, 4)$. We can also make the C^6 norm of $g(t) - g^*$ sufficiently small for all $t \geq \alpha(n)$ such that the Sobolev constant of $g(t)$ is bounded above by $2C(\frac{1}{3}v_0, 3D_0, 4)$ for all $t \geq \alpha(n)$. A desired logarithmic Sobolev inquality then follows for $t \in [\alpha(n), T)$.

A desired logarithmic Sobolev inequality for $g(t)$ on $[0, \alpha(n)]$ follows from Theorem A in [Y1]. It also follows from the arguments for controlling the evolution of the Sobolev constant in [Y3]. ■

References

[CD] B. Chow and D. Knopf, *The Ricci Flow: An Introduction*, Mathematical Surveys and Monographs vol.110, Amer. Math. Soc. , 2004.

[GIK] C. Guenther, J. Isenberg and D. Knopf, *Stability of the Ricci flow at Ricci flat metrics,* Comm. Anal. Geom. **10** (2002), 741-777.

[P] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications,* Nov. 2002, http://arXiv.org/math.DG/0211159v1.

[Y1] R. Ye, *The logarithmic Sobolev inequality along the Ricci flow,* arXiv:0707.2424.

[Y2] R. Ye, *The logarithmic Sobolev inequality along the Ricci flow in dimension 2*, July 2007, arXiv:0708.2003.

[Y3] R. Ye, *Ricci flow, Einstein metrics and space forms*, Tran. Am. Math. Soc. 338 (1993), 871-895.